iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Correlations and susceptibilities in (quasi-)1D disordered spin systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1994 J. Phys. A: Math. Gen. 27 4379

(http://iopscience.iop.org/0305-4470/27/13/014)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.68
The article was downloaded on 01/06/2010 at 21:25

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 27 (1994) 4379–4400. Printed in the UK

Correlations and susceptibilities in (quasi-)1D disordered spin systems

E Amic† and J M Luck‡

Service de Physique Théorique, Centre d'Études de Saclay, 91191 Gif-sur-Yvette cedex, France

Received 12 January 1994

Abstract. We investigate two randomly frustrated 1D or quasi-1D Ising systems with diluted disorder, namely the ferromagnetic chain in a random field and the ladder spin glass. We present an analytical study of the susceptibilities (linear χ, nonlinear χ_3, and higher orders), which characterize the response to a uniform external field at finite temperature. Both models admit a continuum description in the scaling regime of low temperature and low impurity concentration, where the susceptibilities obey power laws, similarly to usual critical phenomena, albeit unlike the mean-field theory of spin glasses. We obtain explicit expressions for the scaling functions of χ and χ_3, and an estimate for the essential Lee–Yang singularity.

1. Introduction

In disordered magnetic systems, such as spin glasses or random-field models, the combined effects of frustration and disorder can generate a complex pattern of low-energy metastable states, with far-reaching physical consequences, such as the appearance of the spin-glass phase, with its unusual thermodynamics, correlations, and dynamics [1, 2].

One of the most celebrated characteristics of the spin-glass transition concerns the nonlinear response to a uniform external field, described by the susceptibilities. The Sherrington–Kirkpatrick mean-field theory [3] predicts that the linear susceptibility χ is finite at the transition temperature T_G, whereas the nonlinear susceptibility diverges as $\chi_3 \sim (T - T_G)^{-\gamma_3}$, with $\gamma_3 = 1$. Experimental observations agree qualitatively with these predictions [2]. We recall the definition of the magnetic susceptibilities, which are the main subject of this article. Consider a model with Ising spins $\{\sigma_i\}$, which obeys spin-flip symmetry on average, such as a spin glass, or a ferromagnet in a quenched random field with an even probability distribution. The free energy $F(H)$ per spin of such a model in a uniform external magnetic field H can be expanded as an even power series of H,

$$F(H) = F(0) + \sum_{\ell=1}^{\infty}(-1)^{\ell}\chi_{2\ell-1}\frac{H^{2\ell}}{2\ell} = F(0) - \chi\frac{H^2}{2} + \chi_3\frac{H^4}{4} + \cdots . \tag{1.1}$$

The magnetization thus reads (for a finite system with N spins)

$$m = \frac{1}{N}\sum_i \langle\sigma_i\rangle = -\frac{\partial F}{\partial H} = \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\chi_{2\ell-1}H^{2\ell-1} = \chi H - \chi_3 H^3 + \cdots . \tag{1.2}$$

† E-mail address: amic@amoco.saclay.cea.fr

‡ E-mail address: luck@amoco.saclay.cea.fr

0305-4470/94/134379+22$19.50 © 1994 IOP Publishing Ltd

The coefficients $\chi_{2\ell-1}$ are the susceptibilities of the model. The *linear susceptibility* χ, which characterizes the linear response of the system, can be expressed as the following sum of the connected two-point correlation function in zero field:

$$\chi = -\left(\frac{\partial^2 F}{\partial H^2}\right)_{H=0} = \frac{\beta}{N}\sum_{i,j}\overline{\langle\sigma_i\sigma_j\rangle = \langle\sigma_i\rangle\langle\sigma_j\rangle} \tag{1.3}$$

whereas the *nonlinear susceptibility* χ_3 can be expressed as a sum of the four-point correlations of the spins, and so on for the *higher-order susceptibilities* $\chi_{2\ell-1}$.

Exactly solvable one-dimensional (1D) disordered magnetic systems provide interesting test cases. In spite of the absence of a finite-temperature phase transition, frustration is responsible for the existence of numerous almost-degenerate states, yielding a rich low-temperature behaviour in thermodynamical quantities and correlation functions. The ferromagnetic Ising chain in a quenched random magnetic field is certainly the simplest of these model systems. It has been the subject of many investigations, based on the observation that its free energy is the Lyapunov exponent of an infinite product of random 2×2 transfer matrices (see [4, 5] for reviews). Its thermodynamical properties at finite temperature are known exactly only for some classes of distributions of the quenched random fields, either discrete [6] or continuous [7–9]. Some results are also available for more general distributions of disorder, either at zero temperature [10], or in the weak-disorder regime [11–13]. The two-point correlation function and the linear susceptibility χ are only known in a limited number of cases [4, 6, 14], whereas virtually nothing is known about higher-order correlations and susceptibilities. The converse situation of an Ising chain with random exchange couplings is not frustrated, but the ladder geometry, i.e. two chains coupled by transversal exchange couplings, is frustrated, and thus provides an interesting spin-glass model [10, 15].

The purpose of the present article is a detailed study of the higher-order susceptibilities of 1D disordered magnetic models, and particularly a comparison of their scaling behaviour with the cases of common critical phenomena and of the mean-field theory of spin glasses. The analytical investigation of the susceptibilities is performed on the 1D disordered Ising models mentioned above, namely the ferromagnetic chain in a random magnetic field (in section 2), and the ladder spin glass (in section 3). The results are put in a more general framework in the discussion of section 4. We restrict ourselves to the case of *diluted disorder*, generated by infinitely strong impurities. This kind of disorder, introduced by Grinstein and Mukamel [6] for the random-field Ising chain, will be shown to allow for exact calculations, even in the more complicated situation of the ladder spin glass. In both models, the most interesting regime is that of low temperature and low impurity concentration, where both the thermal correlation length and the typical distance between two impurities are much larger than the atomic spacing.

2. The Ising chain with infinite random fields

The Hamiltonian of the ferromagnetic Ising chain in a random magnetic field reads

$$\mathcal{H} = -\sum_n (J\sigma_n\sigma_{n+1} + h_n\sigma_n) \tag{2.1}$$

with $J > 0$ being a uniform exchange coupling, and where the h_n are quenched random magnetic fields, with a common even distribution $\rho(h)$.

In this section we restrict the analysis to the limiting situation where all the non-zero random fields are infinitely strong, i.e.

$$h_n = \begin{cases} +\infty & \text{with probability} \quad p/2 \\ 0 & \text{with probability} \quad r = 1 - p \\ -\infty & \text{with probability} \quad p/2 . \end{cases} \tag{2.2}$$

In this limit, all the *impurity spins* which are submitted to random fields $h_n = \pm\infty$ are pinned by these local fields, i.e. frozen. The chain is thus split into a collection of independent finite *clusters*, with prescribed boundary conditions. The spins inside each cluster are free, i.e. they are not submitted to any random field. As a consequence, physical quantities can be evaluated exactly by means of an enumeration procedure. Grinstein and Mukamel [6] have calculated for the first time the thermodynamical properties and the two-point correlation functions of this model at finite temperature, in zero external field. A more detailed analysis has been performed since then, particularly at low temperature [4, 9, 14].

2.1. The free energy in a uniform field

The enumeration approach initiated in [6] can be generalized in order to evaluate the free energy in a uniform external field H. Let $Z_N^{\varepsilon_1,\varepsilon_2}$ be the constrained partition function of a cluster of $(N-1)$ free spins $\{\sigma_1, \ldots, \sigma_{N-1}\}$, with the fixed boundary conditions $\sigma_0 = \varepsilon_1 = \pm 1$ and $\sigma_N = \varepsilon_2 = \pm 1$. The statistical weight of such a cluster is $(p^2/4)r^{N-1}$, so that the free energy per spin at temperature $T = 1/\beta$ reads

$$\beta F' = -\frac{p^2}{4} \sum_{N=1}^{\infty} r^{N-1} \ln\left(Z_N^{++} Z_N^{+-} Z_N^{-+} Z_N^{--}\right) \tag{2.3}$$

where the prime means that the infinite, temperature- and field-independent, contribution of the impurity spins has been omitted.

The transfer-matrix formalism allows to compute the partition functions explicitly. Indeed, we have

$$Z_N^{\varepsilon_1,\varepsilon_2} = \left[(\mathbf{T}_J \mathbf{T}_H)^{N-1} \mathbf{T}_J\right]_{\varepsilon_1,\varepsilon_2} \tag{2.4}$$

where the transfer matrices $\mathbf{T}_J$ and $\mathbf{T}_H$, associated, respectively, with a bond and a site, read

$$\mathbf{T}_J = \begin{pmatrix} e^{\beta J} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J} \end{pmatrix} \qquad \mathbf{T}_H = \begin{pmatrix} e^{\beta H} & 0 \\ 0 & e^{-\beta H} \end{pmatrix} . \tag{2.5}$$

We obtain after some algebra the following expression:

$$\beta F' = -\frac{p^2}{4} \sum_{N=1}^{\infty} r^{N-1} \left[\ln \frac{\left(\lambda_+^N - \lambda_-^N\right)^2}{4W^2} + \ln\left(\left(\lambda_+ \lambda_-\right)^N + \frac{\left(\lambda_+^N - \lambda_-^N\right)^2}{4W^2} \right) \right] \tag{2.6}$$

for the free energy at finite temperature and external field. We have introduced the notation

$$W = \left[1 + e^{4\beta J} \sinh^2(\beta H)\right]^{1/2} \tag{2.7}$$

and

$$\lambda_\pm = e^{\beta J} \cosh(\beta H) \pm e^{-\beta J} W \tag{2.8}$$

are the eigenvalues of the full transfer matrix $\mathbf{T} = \mathbf{T}_J \mathbf{T}_H$ of the pure Ising chain in an external field.

2.2. *A summary of zero-field thermodynamics*

The thermodynamics of the model in the absence of an external field have been studied at length [4, 6, 9]. The expression (2.6) of the free energy simplifies to

$$\beta F' = -r \ln 2 + \tfrac{1}{2} \ln\left(1 - \tau^2\right) - \frac{p^2}{2} \sum_{N=1}^{\infty} r^{N-1} \ln\left(1 - \tau^{2N}\right) \tag{2.9}$$

with the notation

$$\tau = \mathrm{e}^{-2\mu} = \tanh(\beta J)\,. \tag{2.10}$$

The low-temperature behaviour of the result (2.9) can be derived by expanding the generic term of the sum as

$$\ln(1 - \tau^{2N}) = -2\beta J + \ln(4N) - 2N\,\mathrm{e}^{-2\beta J} + \frac{2N^2+1}{3}\,\mathrm{e}^{-4\beta J} + \mathcal{O}(\mathrm{e}^{-6\beta J})\,. \tag{2.11}$$

We thus obtain

$$F' = E_0' - TS_0 - B\,T\,\mathrm{e}^{-4\beta J} + \cdots \tag{2.12}$$

with

$$E_0' = -rJ \qquad S_0 = \frac{p^2}{2} \sum_{N=1}^{\infty} r^{N-1} \ln N \qquad B = \frac{r(4-p)}{6p}\,. \tag{2.13}$$

The expressions for the *ground-state energy* E_0' and for the *zero-temperature entropy* S_0 can be interpreted by considering the *frustrated clusters*, i.e. those bordered by two impurity spins of opposite signs. On such a cluster with $(N-1)$ free spins, there are N ways of placing a Bloch wall, which costs an energy $2J$. The *specific heat* is exponentially small at low temperatures, namely

$$C \approx B(4\beta J)^2\,\mathrm{e}^{-4\beta J}\,. \tag{2.14}$$

2.3. *Zero-temperature properties in a field*

We now investigate the zero-temperature thermodynamical properties of the model in an external field H, i.e. its ground-state energy E_0' and its zero-temperature residual entropy S_0. We take $H > 0$ for definiteness.

The quantities defined in (2.7) and (2.8) have the following low-temperature behaviour $\lambda_\pm \approx \mathrm{e}^{\beta J \pm \beta H}$, $W \approx \mathrm{e}^{\beta(2J+H)}/2$, up to exponentially small corrections, so that (2.6) yields

$$E_0' = -p^2 \sum_{N=1}^{\infty} r^{N-1} \left[(N-1)H + (N-2)J + \max\left\{ J - (N-1)\frac{H}{2}; 0 \right\} \right]. \tag{2.15}$$

The cluster size N thus has to be compared with a fixed integer M, depending on the external field H, whose interpretation will become clear in a while. More precisely, we are led to set

$$\frac{2J}{H} = M - 1 + \theta \tag{2.16}$$

with $M \geqslant 1$ integer, and $0 \leqslant \theta < 1$. Such a decomposition also shows up in the study of the Ising chain in a random field with a symmetric binary distribution ($h_n = \pm H_{\rm B}$) [4, 9].

The ground-state energy reads

$$E_0' = -J\big(r - p\,r^M\big) - \frac{H}{2}\left[r + \big(1 + (M-1)p\big)r^M\right] \qquad (2.17)$$

whereas S_0 vanishes for generic values of H, i.e. for $\theta \neq 0$.

The model therefore has an infinity of pure phases at zero temperature, labelled by the integer M, and separated by first-order transitions. The constant values of the magnetization

$$m_M = \tfrac{1}{2}\left[r + \big(1 + (M-1)p\big)r^M\right] \qquad (2.18)$$

can be viewed as the order parameters of these phases. We have $m_1 = r$ for $M = 1$, since every free spin is aligned with the external field, if the latter is strong enough ($H > 2J$). Conversely, the magnetizations converge to the low-field limit $m_\infty = r/2$. Indeed, all the free spins are aligned with an infinitesimally positive external field, except those which belong to clusters bordered by two negative impurity spins.

For the transition values of the field, such that $2J/H = M$ is exactly an integer, we have

$$E_0' = -rJ\,\frac{r^M + M + 1}{M} \qquad S_0 = \frac{p^2}{4} r^M \ln 2\,. \qquad (2.19)$$

The zero-temperature entropy originates in the clusters of M free spins between two negative impurity spins, on which the two configurations where all the free spins are parallel are degenerate for $2J = MH$ exactly. This is the only possible case of degeneracy in a non-zero external field.

2.4. Scaling regime at low temperature

At low temperatures, the following three length scales show up in a natural fashion.

- The correlation length of the pure ferromagnetic Ising chain in zero field reads $\xi = 1/(2\mu)$, with the notation (2.10), and thus diverges at low temperatures.
- When the concentration p of impurities is small, there is a second diverging length, namely the typical distance $L = 1/p$ between two impurities.
- If the external field H is very weak, we can define a magnetic length $L_H = T/H$, corresponding to the scale on which the interaction energy of an array of parallel spins with the external field becomes comparable to the thermal energy T.

It can therefore be expected that, in the scaling regime where p, T, and H simultaneously go to zero, the system admits a continuum description, where the three divergent lengths have to be compared, and where physical quantities keep a dependence in the dimensionless ratios of these lengths. We choose to work with the following scaling variables:

$$x = \frac{2\mu}{p} = \frac{L}{\xi} \qquad y = \frac{\beta H}{p} = \frac{L}{L_H}\,. \qquad (2.20)$$

The occurrence of a scaling behaviour, when p, T, and H go to zero, with fixed values of the scaling variables x and y, can be checked explicitly in the case of the free energy by estimating the behaviour of (2.6) in the scaling regime. We thus obtain

$$F' \approx E_0' - TS_0 - \frac{pT}{2}\Phi(x, y) \qquad (2.21)$$

where the zero-field ground-state energy E_0' and zero-temperature entropy S_0 have been given in (2.13).

The scaling function $\Phi(x, y)$ describes the contribution to the free energy of the low-temperature, long-wavelength excitations of the system. It admits the following integral representation:

$$\Phi(x, y) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \left[\ln \frac{\sinh(tw/2)}{tw/2} + \frac{1}{2} \ln \left(1 + \frac{x^2}{w^2} \sinh^2(tw/2) \right) \right] \qquad (2.22)$$

with the notation

$$w = \left(x^2 + 4y^2\right)^{1/2}. \qquad (2.23)$$

The ratio w/x is nothing but the scaling form of the quantity W, defined in (2.7). An integration by parts yields the alternative expression

$$\Phi(x, y) = \frac{w}{2} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \left[\coth(tw/2) - \frac{2}{tw} + \frac{\sinh(tw)}{\cosh(tw) - 1 + 2w^2/x^2} \right]. \qquad (2.24)$$

Several limiting situations deserve our attention.

2.4.1. Zero magnetic field: scaling in the p–T plane. In the absence of an external magnetic field ($y = 0$, whence $w = x$), the scaling regime in the p–T plane is described by the one-variable scaling function

$$\Phi(x, 0) = \mathcal{F}(x) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \ln \frac{\sinh(tx)}{tx} = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \left[x \coth(tx) - \frac{1}{t} \right] \qquad (2.25)$$

a result known previously [4, 9].

The scaling function $\mathcal{F}(x)$ has been shown to play a role in the scaling analysis of other disordered systems, such as the conductance of electrical ladders [16], the 2D Ising model with layered randomness [17, 18], and the quantum Ising chain with random exchanges in a transverse field [19]. The function $\mathcal{F}(x)$, defined by (2.25) for $\mathrm{Re}\, x > 0$, can be rewritten as

$$\mathcal{F}(x) = -\psi\left(\frac{1}{2x}\right) - x - \ln(2x) = \sum_{\ell=1}^{\infty} \frac{B_{2\ell}}{2\ell} (2x)^{2\ell}. \qquad (2.26)$$

In these expressions, ψ is the digamma function, the logarithmic derivative of Euler's Γ-function, whereas

$$B_{2\ell} = \frac{(-1)^{\ell-1} 2\, (2\ell)!\, \zeta(2\ell)}{(2\pi)^{2\ell}} \qquad (2.27)$$

are the Bernoulli numbers, related to Riemann's ζ-function.

The behaviour of the scaling function for small and large values of x is the following:

$$\mathcal{F}(x) = \tfrac{1}{3}x^2 - \tfrac{2}{15}x^4 + \tfrac{16}{63}x^6 + \cdots \qquad x \to 0 \qquad (2.28a)$$

$$\mathcal{F}(x) = x - \ln(2x) + \gamma_{\mathrm{E}} + \cdots \qquad x \to \infty \qquad (2.28b)$$

where γ_{E} is Euler's constant.

The function $\mathcal{F}(x)$ is indefinitely differentiable, but not analytic at $x = 0$. In other terms, the power series (2.26) is divergent, and its sum has an exponentially small non-perturbative ambiguity, or singular part, near $x = 0$. In order to give a precise meaning to this last statement, and along the lines of [19], it is advantageous to extend $\mathcal{F}(x)$ to an even function of x, by setting $\mathcal{F}(x) = \mathcal{F}(-x)$ for $\mathrm{Re}\, x < 0$. The function thus obtained has a discontinuity along the imaginary axis, i.e.

$$\mathcal{F}_{\mathrm{sg}}(|x|) = \mathcal{F}(\mathrm{i}|x| + 0) - \mathcal{F}(\mathrm{i}|x| - 0) = \frac{2\pi \mathrm{i}}{\mathrm{e}^{\pi/|x|} - 1} . \qquad (2.29)$$

This exact result is a consequence of the difference identity for the ψ-function, namely $\psi(z) - \psi(-z) = -\pi \cot(\pi z) - 1/z$.

The singular part of $\mathcal{F}$ can be estimated in the following alternative way. The difference considered in (2.29) can be expressed by deforming the second representation of (2.25) to a complex contour integral which encircles the poles of the integrand. Since the nearest pole lies at $t = \mathrm{i}\pi/x$, we get the exponential estimate

$$\mathcal{F}_{\mathrm{sg}} \sim \mathrm{e}^{-\pi/|x|} . \qquad (2.30)$$

This result can alternatively be related to the large-order behaviour of the coefficients of the power series (2.26) by using the asymptotic form of the Bernoulli numbers (2.27) along the lines of a resummation procedure for divergent series, used extensively e.g. in quantum field theory (see e.g. [20]).

2.4.2. Zero temperature: scaling in the p–H plane. The scaling regime in the p–H plane is described by the scaling function

$$\Phi(0, y) = \mathcal{F}(y) . \qquad (2.31)$$

We thus obtain the same functional form in x at $y = 0$, and in y at $x = 0$.

The small-y behaviour of $\Phi(0, y)$, related to the susceptibilities, will be discussed in detail in section 2.5. Its large-y behaviour matches the small-H behaviour of the ground-state energy (2.17), namely $E_0' = -J - H/2 + \mathcal{O}(p)$.

2.4.3. Weak disorder: scaling in the T–H plane. The limit where both scaling variables x and y are simultaneously large describes the crossover to the scaling behaviour of the pure Ising chain in the T–H plane. An evaluation of the integral representation (2.24) yields to

$$\Phi(x, y) = w - \ln x + \gamma_{\mathrm{E}} + \mathcal{O}(1/x,\ 1/y) . \qquad (2.32)$$

The leading term of this expansion is in accord with the singular part of the free energy of the pure chain in the scaling regime, i.e.

$$\beta F_{\mathrm{sg}} = \beta(F' + J) \approx -(\beta^2 H^2 + \mu^2)^{1/2} = -pw/2 . \qquad (2.33)$$

This estimate can be recast as $\beta F_{\mathrm{sg}} \approx 1/\mathcal{L}$, where $\mathcal{L}$ is the following combination of the magnetic and thermal lengths:

$$\frac{1}{\mathcal{L}^2} = \frac{1}{L_H^2} + \frac{4}{\xi^2} . \qquad (2.34)$$

2.4.4. Complex magnetic field: the Lee–Yang singularities. The analytic structure of the two-variable scaling function in the complex y-plane is also worth being investigated, since it yields the form of the Lee–Yang singularities [21, 22] of the present problem.

The free energy is expected to be analytic around $H = 0$ at finite temperature. The scaling function $\Phi(x, y)$ is indeed analytic around $y = 0$, for any $x \neq 0$. It has isolated essential singularities at the complex points $y = \pm \mathrm{i}x/2$, where the variable w of (2.23) vanishes. The position of the Lee–Yang singularities is therefore the same in the pure Ising chain and in the random one. The asymptotic form of these singularities as $w \to 0$ is again governed by the position of the nearest pole of the integrand in (2.24), which lies close to $t = 2\mathrm{i}\pi/w$ for $|w|$ small, whence the estimate

$$\Phi_{\mathrm{sg}} \sim \mathrm{e}^{-2\pi/|w|} \tag{2.35}$$

which matches (2.30) at very low temperature ($x = 0$).

The result (2.35) can be recast as $\Phi_{\mathrm{sg}} \sim r^{\pi|\mathcal{L}|}$, where $\mathcal{L}$ is the length introduced in (2.34). In other words, the Lee–Yang singular free energy is approximately the statistical weight of a cluster whose length $\mathcal{L}$ scales as the reciprocal singular free energy of the pure model.

2.5. The susceptibilities

2.5.1. Finite temperature. Let us now turn to the analysis of the susceptibilities of the random-field chain, with emphasis on χ and χ_3. By expanding (2.6) as a power series in H, we can obtain, at least in principle, all the susceptibilities at finite temperature.

The linear susceptibility reads

$$\chi = \beta p^2 \frac{1+\tau}{1-\tau} \sum_{N=1}^{\infty} r^{N-1} \frac{1+\tau^{2N}}{1+\tau^N} \left[\frac{N}{1-\tau^N} - \frac{1+\tau}{(1-\tau)(1+\tau^N)} \right]. \tag{2.36}$$

By expanding the denominators first, this expression can be recast in the following form, which exhibits its structure in the complex temperature plane:

$$\chi = \beta p^2 r (1-\tau^2)^2 \sum_{a=0}^{\infty} \frac{(2a+1)\tau^{4a}(1+r\tau^{2a+1})}{(1-r\tau^{2a})^2(1-r\tau^{2a+1})(1-r\tau^{2a+2})^2}. \tag{2.37}$$

These results are in agreement with those of [4, 14]. Let us notice that in [14] χ is defined from (1.3) without the prefactor β.

We obtain a formula similar to (2.36) for the nonlinear susceptibility, namely

$$\chi_3 = \beta^3 p^2 \frac{1+\tau}{3(1-\tau)^4} \sum_{N=1}^{\infty} r^{N-1} \frac{A_N N^2 + B_N N + C_N}{(1-\tau^N)^2(1+\tau^N)^4} \tag{2.38}$$

where A_N, B_N, and C_N are the following polynomials in τ and τ^N:

$$\begin{aligned} A_N &= 12\tau^{2N}(1-\tau)^2(1+\tau)(1+\tau^N)^2 \\ B_N &= (1-\tau)(1-\tau^{2N})\big[(1+4\tau+\tau^2)(1+\tau^{4N}) + 2(7+16\tau+7\tau^2)\tau^N(1+\tau^{2N}) \\ &\qquad - 2(11+20\tau+11\tau^2)\tau^{2N}\big] \\ C_N &= -(1+\tau)(1-\tau^N)^2\big[(1+10\tau+\tau^2)(1+\tau^{4N}) + 8(1+4\tau+\tau^2)\tau^N(1+\tau^{2N}) \\ &\qquad - 2(5+2\tau+5\tau^2)\tau^{2N}\big]. \end{aligned} \tag{2.39}$$

2.5.2. Scaling regime at low temperature. The most interesting situation is again the low-temperature scaling regime, where the scaling function $\Phi(x, y)$ is the generating function of all the susceptibilities. To be more specific, if we introduce the notation

$$\Phi(x, y) = \mathcal{F}(x) + \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} S_{2\ell-1}(x) y^{2\ell} = \mathcal{F}(x) + S(x) y^2 - \tfrac{1}{2} S_3(x) y^4 + \cdots \qquad (2.40)$$

we obtain the following scaling laws:

$$\chi = \frac{\beta}{p} S(x) \qquad \chi_3 = \left(\frac{\beta}{p}\right)^3 S_3(x) \quad \cdots \quad \chi_{2\ell-1} = \left(\frac{\beta}{p}\right)^{2\ell-1} S_{2\ell-1}(x) \quad \cdots . \qquad (2.41)$$

The scaling function of the linear susceptibility is obtained by either expanding the expression (2.24) of $\Phi(x, y)$, or by estimating the behaviour of the result (2.37) in the scaling regime. It reads

$$S(x) = \frac{2}{x} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t}\, t \coth(tx) - \frac{2}{x^2} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \frac{\cosh(tx)}{\cosh^2(tx/2)} . \qquad (2.42)$$

This result can again be expressed in terms of the digamma function ψ, and of its first derivative

$$S(x) = \frac{1}{x^3} \psi' \left(\frac{1}{2x}\right) + \frac{8}{x^4} \left[\psi\left(\frac{1}{x}\right) - \psi\left(\frac{1}{2x}\right) - \ln 2\right] - \frac{4}{x^3} - \frac{4}{x^2} - \frac{2}{x} . \qquad (2.43)$$

The asymptotic expansions of $S(x)$ are as follows:

$$S(x) = \tfrac{1}{3} + \tfrac{14}{15} x^2 - \tfrac{229}{42} x^4 + \cdots \qquad x \to 0 \qquad (2.44a)$$

$$S(x) = \frac{2}{x} - \frac{4}{x^2} + \frac{4 + \pi^2/6}{x^3} + \cdots \qquad x \to \infty . \qquad (2.44b)$$

The low-temperature susceptibility can be derived from (2.36) for any value of the dilution p,

$$\chi = \frac{\beta p^2}{6} \sum_{N=1}^{\infty} r^{N-1} (N^2 - 1) = \frac{\beta r (2 + p)}{6p} \approx \frac{\beta}{3p} \qquad p \to 0 \qquad (2.45)$$

the small-p behaviour being in accord with the value $S(0) = \frac{1}{3}$. Conversely, the large-x behaviour $S(x) \approx 2/x$ describes the crossover for $p \ll \mu$ from the disordered chain to the ordered one. The susceptibility of the latter reads $\chi = \beta\, \mathrm{e}^{2\beta J} \approx \beta/\mu$.

Similarly, the scaling function of the nonlinear susceptibility reads

$$S_3(x) = \frac{4}{x^2} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \frac{t^2}{\sinh^2(tx)} + \frac{1}{x^3} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t}\, t \frac{\cosh(2tx) + 10\cosh(tx) - 7}{\sinh(tx)\cosh^2(tx/2)}$$
$$- \frac{2}{x^4} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t} \frac{\cosh(2tx) + 4\cosh(tx) - 1}{\cosh^4(tx/2)} . \qquad (2.46)$$

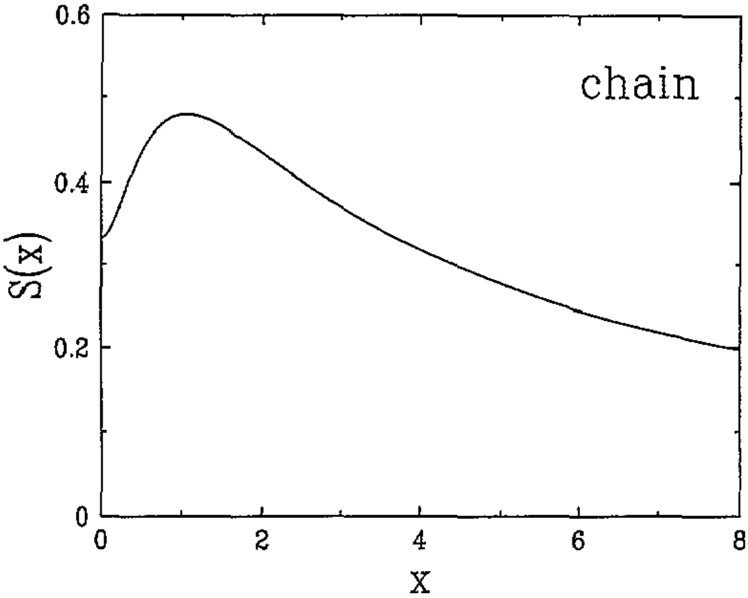

Figure 1. Plot of the scaling function $S(x)$ of the linear susceptibility χ of the Ising chain with infinitely strong random fields.

This function now involves the digamma function ψ and its first two derivatives, namely

$$S_3(x) = \frac{1}{x^3}\left\{\frac{1}{x^3}\psi''\left(\frac{1}{2x}\right) + \frac{16}{x^4}\left[2\psi'\left(\frac{1}{x}\right) - \psi'\left(\frac{1}{2x}\right)\right] + \frac{6}{x^2}\psi'\left(\frac{1}{2x}\right)\right.$$
$$+\frac{64}{3x^3}\left(4-\frac{1}{x^2}\right)\left[\psi\left(\frac{1}{x}\right) - \psi\left(\frac{1}{2x}\right) - \ln 2\right]$$
$$\left.+\frac{32}{3x^4} + \frac{16}{3x^3} - \frac{80}{3x^2} - \frac{16}{x} - 4\right\}. \qquad (2.47)$$

The asymptotic expansions of $S_3(x)$ are as follows:

$$S_3(x) = \tfrac{4}{15} - \tfrac{176}{21}x^2 + \cdots \qquad x \to 0 \qquad (2.48a)$$

$$S_3(x) = \frac{4}{x^3} - \frac{16}{x^4} + \cdots \qquad x \to \infty. \qquad (2.48b)$$

The low-temperature nonlinear susceptibility can alternatively be derived from (2.38) for any value of the dilution p

$$\chi_3 = \frac{\beta^3 p^2}{90}\sum_{N=1}^{\infty} r^{N-1}(N^4 - 1) = \frac{\beta^3 r(24 - 12p + 2p^2 + p^3)}{90p^3} \approx \frac{4\beta^3}{15p^3} \qquad p \to 0 \quad (2.49)$$

the small-p behaviour being in accord with the value $S_3(0) = \frac{4}{15}$. Conversely, the large-x behaviour $S_3(x) \approx 4/x^3$ describes the crossover for $p \ll \mu$ from the disordered chain to the ordered one. The nonlinear susceptibility of the latter reads $\chi_3 = \beta^3 \mathrm{e}^{2\beta J}(3\,\mathrm{e}^{4\beta J} - 1)/6 \approx \beta^3/(2\mu^3)$.

Figures 1 and 2 show plots of the scaling functions $S(x)$ and $S_3(x)$.

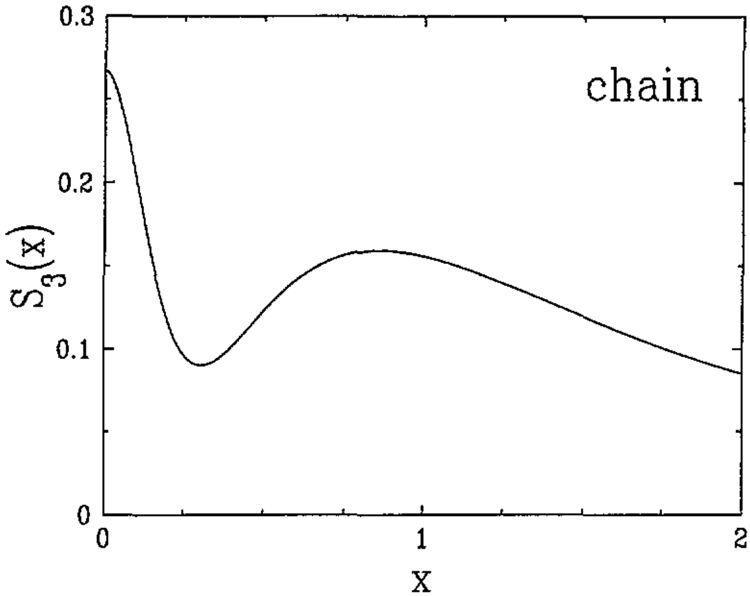

Figure 2. Plot of the scaling function $S_3(x)$ of the nonlinear susceptibility χ_3 of the Ising chain with infinitely strong random fields.

We end up this section with a few general properties of the scaling functions $S_{2\ell-1}(x)$ of the higher-order susceptibilities $\chi_{2\ell-1}$. First, their values at $x = 0$ can be obtained from the expression (2.31) of $\Phi(0, y)$ and from the expansion (2.26)

$$S_{2\ell-1}(0) = (-1)^{\ell-1} 2^{2\ell-1} B_{2\ell} = \frac{(2\ell)!\,\zeta(2\ell)}{\pi^{2\ell}}. \tag{2.50}$$

On the other hand, the low-temperature behaviour of the higher-order susceptibilities of the pure ferromagnetic Ising chain can be derived by expanding the estimate (2.33), namely

$$\chi_{2\ell-1} \approx \frac{(2\ell-3)!!}{2^{\ell-1}(\ell-1)!}\left(\frac{\beta}{\mu}\right)^{2\ell-1}. \tag{2.51}$$

We thus obtain the large-x fall-off of the scaling functions

$$S_{2\ell-1}(x) \approx \frac{2^{\ell}(2\ell-3)!!}{(\ell-1)!}\, x^{-(2\ell-1)} \qquad x \to \infty. \tag{2.52}$$

Finally, the general analytic structure of the scaling functions of the higher-order susceptibilities is clearly visible on (2.43) and (2.47): $S_{2\ell-1}(x)$ is a linear combination with rational coefficients of the digamma function, and of its first ℓ derivatives, up to $\psi^{(\ell)}(x)$.

3. The ladder spin glass

3.1. The model

We now consider a ladder-shaped spin glass [10, 15], consisting of two interacting Ising chains. The model is defined by the Hamiltonian

$$\mathcal{H} = -\sum_n \left[J\left(\sigma_n^{(1)}\sigma_{n+1}^{(1)} + \sigma_n^{(2)}\sigma_{n+1}^{(2)}\right) + K_n \sigma_n^{(1)}\sigma_n^{(2)} + H(\sigma_n^{(1)} + \sigma_n^{(2)})\right]. \tag{3.1}$$

There is a uniform ferromagnetic longitudinal exchange coupling $J > 0$ between neighbouring spins along the ladder, and a uniform external magnetic field H, whereas the transversal exchange couplings K_n across the ladder between the spins $\sigma_n^{(1)}$ and $\sigma_n^{(2)}$ of the nth column are quenched random variables. Figure 3 fixes the notations.

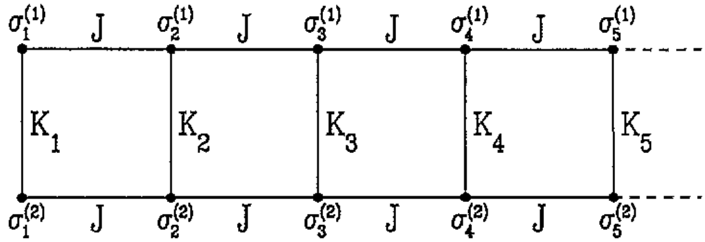

Figure 3. Schema of the ladder spin glass, showing the conventions and notations used for the spins and the exchange couplings.

We restrict again ourselves to the limiting situation where the random exchange interactions are either zero or infinite, according to the law

$$K_n = \begin{cases} +\infty & \text{with probability} \quad p/2 \\ 0 & \text{with probability} \quad r = 1 - p \\ -\infty & \text{with probability} \quad p/2 \,. \end{cases} \tag{3.2}$$

The infinitely large transversal exchange coupling which acts on the pair of spins of every *impurity column* either identifies them ($\sigma_n^{(1)} = \sigma_n^{(2)}$ if $K_n = +\infty$), or forces them to be anti-parallel ($\sigma_n^{(1)} = -\sigma_n^{(2)}$ if $K_n = -\infty$). The values of the individual spins are not determined, so that the impurities do not have the effect of splitting the ladder into independent finite clusters. The present model is thus more difficult than the chain with infinitely strong random fields, studied in section 2. In spite of this difficulty, we have evaluated its zero-field thermodynamics and its linear susceptibility at finite temperature.

We start with some general formalism. Let $Z_n^{\varepsilon_1,\varepsilon_2}$ be the restricted partition function at temperature $T = 1/\beta$ of the first n square cells, or *plaquettes*, in an external field, and with prescribed boundary conditions $\sigma_n^{(1)} = \varepsilon_1$, $\sigma_n^{(2)} = \varepsilon_2$. Because of the up–down symmetry of the ladder, we have $Z_n^{+-} = Z_n^{-+}$. The partition functions of two successive sizes $(n-1)$ and n obey the following recursion relations:

$$Z_n^{++} = \mathrm{e}^{\beta(K_n+2H)}\left(\mathrm{e}^{2\beta J} Z_{n-1}^{++} + 2Z_{n-1}^{+-} + \mathrm{e}^{-2\beta J} Z_{n-1}^{--}\right) \tag{3.3a}$$

$$Z_n^{+-} = \mathrm{e}^{-\beta K_n}\left(Z_{n-1}^{++} + 2c_J Z_{n-1}^{+-} + Z_{n-1}^{--}\right) \tag{3.3b}$$

$$Z_n^{--} = \mathrm{e}^{\beta(K_n-2H)}\left(\mathrm{e}^{-2\beta J} Z_{n-1}^{++} + 2Z_{n-1}^{+-} + \mathrm{e}^{2\beta J} Z_{n-1}^{--}\right) \tag{3.3c}$$

with the notation

$$c_J = \cosh(2\beta J) \qquad s_J = \sinh(2\beta J) \qquad c_H = \cosh(2\beta H) \qquad s_H = \sinh(2\beta H)\,. \tag{3.4}$$

Let us introduce, for convenience, the following parametrization:

$$Z_n^{++} = x_n + y_n \qquad Z_n^{--} = x_n - y_n \qquad Z_n^{+-} = w_n\,. \tag{3.5}$$

In terms of the new variables, the recursion (3.3) reads

$$x_n = 2\,\mathrm{e}^{\beta K_n}\big(c_J c_H x_{n-1} + c_H w_{n-1} + s_J s_H y_{n-1}\big) \tag{3.6a}$$

$$y_n = 2\,\mathrm{e}^{\beta K_n}\big(c_J s_H x_{n-1} + s_H w_{n-1} + s_J c_H y_{n-1}\big) \tag{3.6b}$$

$$w_n = 2\,\mathrm{e}^{-\beta K_n}\big(x_{n-1} + c_J w_{n-1}\big)\,. \tag{3.6c}$$

3.2. *The free energy in zero field*

When the external magnetic field vanishes, the variables y_n vanish because the system recovers the spin-flip symmetry, and the recursion relations (3.6) assume the form

$$x_n = 2\,\mathrm{e}^{\beta K_n}(c_J x_{n-1} + w_{n-1}) \tag{3.7a}$$

$$w_n = 2\,\mathrm{e}^{-\beta K_n}(x_{n-1} + c_J w_{n-1})\,. \tag{3.7b}$$

As already noticed in [15], this problem is equivalent to that of a random-field Ising chain, up to the substitution $K_n \to h_n$, $c_J \to \mathrm{e}^{2\beta J}$. The free energy of the model *per plaquette* is given by the Lyapunov exponent of the infinite product $\prod_{n=1}^{\infty} \mathcal{T}_n$, where $\mathcal{T}_n$ is the 2×2 transfer matrix defined by the linear recursion (3.7). We follow from here on the Dyson–Schmidt approach (see [4] for a review). We introduce the ratios, or Riccati variables, $\rho_n = x_n/w_n$. These quantities obey a recursive map of the form

$$\rho_n = \mathrm{e}^{2\beta K_n}\frac{c_J\rho_{n-1} + 1}{\rho_{n-1} + c_J}\,. \tag{3.8}$$

When the label n becomes large, the Riccati variables ρ_n admit a stationary limit distribution, which is invariant under the transform (3.8). The free energy is then given by

$$-\beta F = \ln 2 - \beta\overline{K} + \big\langle \ln(\rho_n + c_J)\big\rangle \tag{3.9}$$

where the brackets $\langle\cdots\rangle$ denote an average WRT the invariant measure of the ρ_n.

For the sake of convenience, and along the lines of [4, 7, 9, 14], we introduce the variables

$$z_n^{(0)} = \frac{1-\rho_n}{1+\rho_n} = \frac{w_n - x_n}{w_n + x_n}\,. \tag{3.10}$$

In terms of these variables, and for the distribution (3.2) of the random exchange couplings, the recursion relations (3.7) and (3.8) take the form

$$z_n^{(0)} = \begin{cases} +1 & \text{if} \quad K_n = -\infty \\ \tau^2 z_{n-1}^{(0)} & \text{if} \quad K_n = 0 \\ -1 & \text{if} \quad K_n = +\infty \end{cases} \tag{3.11}$$

where τ has been introduced in (2.10). The invariant probability density of the $z_n^{(0)}$ thus reads

$$\mu(z) = \frac{p}{2}\sum_{k=0}^{\infty} r^k[\delta(z-\tau^{2k}) + \delta(z+\tau^{2k})]\,. \tag{3.12}$$

Furthermore, the expression (3.9) of the free energy becomes

$$-\beta F = 2\ln 2 - \ln(1-\tau^2) + \left\langle \ln \frac{1+\tau^2 z_n^{(0)}}{1+z_n^{(0)}} \right\rangle . \tag{3.13}$$

In order to obtain a well defined finite value of the free energy of the model, one has to subtract the infinite, temperature-independent, contribution of the infinite transversal couplings $K_n = \pm\infty$. It turns out that the term to be subtracted coincides with the divergent contribution of the value $z_n^{(0)} = -1$ to the average in (3.13). We have indeed, assuming for a while that the transversal exchange couplings are not strictly infinite ($K_n = \pm K$, with $K \to \infty$)

$$z_n^{(0)} = -1 + 2\,\frac{1+\tau^2 z_{n-1}^{(0)}}{1-\tau^2 z_{n-1}^{(0)}}\,\mathrm{e}^{-2\beta K} + \mathcal{O}\big(\mathrm{e}^{-4\beta K}\big) . \tag{3.14}$$

The finite part of the free energy, i.e.

$$F' = \lim_{K\to\infty} (F + pK) \tag{3.15}$$

is thus given by

$$\beta F' = -(1+r)\ln 2 + \ln(1-\tau^2) - \frac{p^2}{2}\sum_{k=1}^{\infty} r^{k-1}\ln(1-\tau^{4k}) . \tag{3.16}$$

This result is very similar to (2.9). Its low-temperature behaviour follows (2.12) and (2.14), with

$$E_0' = -(1+r)J \qquad S_0 = \frac{p^2}{2}\sum_{k=1}^{\infty} r^{k-1}\ln(2k) \qquad B = \frac{16-14p+p^2}{6p} . \tag{3.17}$$

The ground-state energy E_0' shows an excess of pJ with respect to two decoupled Ising chains, due to the *frustrated clusters*, namely sequences of plaquettes bordered by two infinite transversal couplings of opposite signs. These clusters have a frequency $p/2$. Every such cluster causes an excess of energy of $2J$, since there has to be one Bloch wall on either chain inside the cluster. The zero-temperature entropy S_0 reflects that there are $2k$ different ways of placing the Bloch wall inside a frustrated cluster of length k. The exponent $4J$ which governs the low-temperature behaviour of the specific heat is nothing but the energy gap between the ground states and the lowest excitations; the latter correspond to flipping any number of spins on either chain within a cluster.

Finally, the free energy of the ladder spin glass exhibits the very same kind of scaling behaviour as that of the random-field chain, when T and p are both small, namely

$$F' \approx E_0' - TS_0 - \frac{pT}{2}\mathcal{F}(2x) \tag{3.18}$$

where the variable x and the function $\mathcal{F}$ have been defined in (2.20) and (2.25), respectively.

3.3. The linear susceptibility at finite temperature

We now turn to the evaluation of the linear susceptibility χ of the ladder spin glass at finite temperature. To do so, we have to expand in a systematic way the variables which enter (3.3) and (3.6), up to second order in the external field H. The lengthy derivations which follow lead to the explicit final formulae (3.37) and (3.38).

It is advantageous to introduce the following notations, for any non-zero field:

$$z_n = \frac{w_n - x_n}{w_n + x_n} \qquad S_n = \mathrm{e}^{-2\beta K_n} \frac{y_n}{w_n}. \tag{3.19}$$

These variables obey the recursion relations

$$z_n = \frac{\mathrm{e}^{-2\beta K_n}(1+\tau^2 z_{n-1}) - c_H(1-\tau^2 z_{n-1}) - \tau s_H\,\mathrm{e}^{2\beta K_{n-1}} S_{n-1}(1+z_{n-1})}{\mathrm{e}^{-2\beta K_n}(1+\tau^2 z_{n-1}) + c_H(1-\tau^2 z_{n-1}) + \tau s_H\,\mathrm{e}^{2\beta K_{n-1}} S_{n-1}(1+z_{n-1})} \tag{3.20a}$$

$$S_n = \frac{s_H(1-\tau^2 z_{n-1}) + \tau c_H\,\mathrm{e}^{2\beta K_{n-1}} S_{n-1}(1+z_{n-1})}{1+\tau^2 z_{n-1}} \tag{3.20b}$$

with the notation given in (3.4).

The observation that the z_n (respectively, the S_n) are even (respectively, odd) functions of the external field H leads to the expansions

$$z_n = z_n^{(0)} - (2\beta H)^2 z_n^{(2)} + \cdots \tag{3.21a}$$

$$S_n = 2\beta H\, S_n^{(1)} + \cdots. \tag{3.21b}$$

In terms of these H-independent variables, the recursion formulae take the following form

$$z_n^{(0)} = \frac{\mathrm{e}^{-2\beta K_n}\left(1+\tau^2 z_{n-1}^{(0)}\right) - \left(1-\tau^2 z_{n-1}^{(0)}\right)}{\mathrm{e}^{-2\beta K_n}\left(1+\tau^2 z_{n-1}^{(0)}\right) + \left(1-\tau^2 z_{n-1}^{(0)}\right)} \tag{3.22a}$$

$$S_n^{(1)} = \frac{1-\tau^2 z_{n-1}^{(0)} + \tau\,\mathrm{e}^{2\beta K_{n-1}} S_{n-1}^{(1)}\left(1+z_{n-1}^{(0)}\right)}{1+\tau^2 z_{n-1}^{(0)}} \tag{3.22b}$$

$$z_n^{(2)} = \frac{\left(1-\tau^4 z_{n-1}^{(0)2}\right) + 2\tau\,\mathrm{e}^{2\beta K_{n-1}} S_{n-1}^{(1)}\left(1+z_{n-1}^{(0)}\right)\left(1+\tau^2 z_{n-1}^{(0)}\right) + 4\tau^2 z_{n-1}^{(2)}}{\left[\mathrm{e}^{-\beta K_n}\left(1+\tau^2 z_{n-1}^{(0)}\right) + \mathrm{e}^{\beta K_n}\left(1-\tau^2 z_{n-1}^{(0)}\right)\right]^2}. \tag{3.22c}$$

For the distribution (3.2) of the random exchange couplings, (3.22a) coincides with (3.11). Furthermore, the expression (3.13) of the free energy remains valid, up to the substitution $z_n^{(0)} \to z_n$. We thus obtain

$$\chi = 8\beta(1-\tau^2)\left\langle \frac{z_n^{(2)}}{\left(1+z_n^{(0)}\right)\left(1+\tau^2 z_n^{(0)}\right)} \right\rangle \tag{3.23}$$

where $\langle\cdots\rangle$ stands for the invariant joint distribution of the random variables $z_n^{(0)}$ and $z_n^{(2)}$.

Because of the complexity of the recursion relations (3.22), we prefer to split the average in (3.23) into constrained averages, according to the value of K_n, namely

$$\chi = 8\beta(1-\tau^2)\left\{ \frac{p}{2}\left\langle \frac{z_n^{(2)}}{\left(1+z_n^{(0)}\right)\left(1+\tau^2 z_n^{(0)}\right)} \right\rangle_{K_n=+\infty} + r\left\langle \frac{z_n^{(2)}}{\left(1+z_n^{(0)}\right)\left(1+\tau^2 z_n^{(0)}\right)} \right\rangle_{K_n=0} \right\}. \tag{3.24}$$

The third contribution ($K_n = -\infty$) to the susceptibility indeed vanishes, because two opposite spins of the same column give no net change of the magnetization when the external field pins either of these spins.

Setting again for a while $K_n = K \to +\infty$, (3.22c) yields the estimate

$$z_n^{(2)} \approx \mathrm{e}^{-2\beta K} \frac{\left(1-\tau^4 z_{n-1}^{(0)2}\right) + 2\tau\, \mathrm{e}^{2\beta K_{n-1}} S_{n-1}^{(1)} \left(1+z_{n-1}^{(0)}\right)\left(1+\tau^2 z_{n-1}^{(0)}\right) + 4\tau^2 z_{n-1}^{(2)}}{\left(1-\tau^2 z_{n-1}^{(0)}\right)^2} \tag{3.25}$$

so that the first term of (3.24) reads

$$\chi_{(1)} = 2\beta p \left\langle \frac{\left(1-\tau^4 z_{n-1}^{(0)2}\right) + 2\tau\, \mathrm{e}^{2\beta K_{n-1}} S_{n-1}^{(1)} \left(1+z_{n-1}^{(0)}\right)\left(1+\tau^2 z_{n-1}^{(0)}\right) + 4\tau^2 z_n^{(2)}}{1-\tau^4 z_{n-1}^{(0)2}} \right\rangle. \tag{3.26}$$

We thus obtain

$$\chi = 2\beta \left\{ p + \frac{2p^2\tau \mathcal{S}}{1+\tau^2} + 2pr\tau \sum_{k=0}^{\infty} (b_k + b_{k+1})\tau^{2k} + 4pr\tau^2 \sum_{k=0}^{\infty} a_{2k}\tau^{4k} \right.$$
$$\left. + 4r \sum_{k=0}^{\infty} (-1)^k \left(1-\tau^{2k+2}\right) a_k \right\} \tag{3.27}$$

where we have introduced the constrained averages

$$a_k = \left\langle z_n^{(2)} z_n^{(0)k} \right\rangle_{K_n=0} \tag{3.28a}$$

$$b_k = \left\langle S_n^{(1)} z_n^{(0)k} \right\rangle_{K_n=0} \tag{3.28b}$$

$$\mathcal{S} = \left\langle S_n^{(1)} \frac{1+\tau^2 z_{n-1}^{(0)}}{1-\tau^2 z_{n-1}^{(0)}} \right\rangle_{K_n=+\infty} . \tag{3.28c}$$

The evaluation of the latter quantities is still rather lengthy. First, using (3.22b), we obtain

$$\mathcal{S} = 1 + \frac{p\tau}{1+\tau^2}\mathcal{S} + r\tau \left\langle S_n^{(1)} \frac{1+z_n^{(0)}}{1-\tau^2 z_n^{(0)}} \right\rangle_{K_n=0} \tag{3.29}$$

whence

$$r\tau \sum_{k=0}^{\infty} (b_k + b_{k+1})\tau^{2k} = \frac{1-p\tau+\tau^2}{1+\tau^2}\mathcal{S} - 1 . \tag{3.30}$$

We can thus eliminate the b_k and obtain

$$\chi = 2\beta \left\{ 2p\mathcal{S} - p + 4r \left[\sum_{k=0}^{\infty} (1 - r\tau^{4k+2}) a_{2k} - \sum_{k=0}^{\infty} (1-\tau^{4k+4}) a_{2k+1} \right] \right\} . \tag{3.31}$$

Second, we have to determine the a_k and $\mathcal{S}$. If $K_n = 0$, we have

$$S_n^{(1)}\left(1+z_n^{(0)}\right) = 1 - \tau^2 z_{n-1}^{(0)} + \tau\, \mathrm{e}^{2\beta K_{n-1}} S_{n-1}^{(1)} \left(1+z_{n-1}^{(0)}\right) \tag{3.32}$$

so that

$$b_k + b_{k+1} = \frac{\tau^{2k}}{1 - r\tau^{2k+1}}\left[C_k - \tau^2 C_{k+1} + (-1)^k p\tau\mathcal{S}\right] \tag{3.33}$$

where the moments $C_k = \langle z_n^{(0)k}\rangle$ can be evaluated from the distribution (3.12)

$$C_k = \begin{cases} \dfrac{p}{1 - r\tau^{2k}} & k \text{ even} \\ 0 & k \text{ odd} . \end{cases} \tag{3.34}$$

Moreover, observing that $\mathcal{S}$ does not actually depend on the constraint $K_n = +\infty$, (3.33) leads to

$$\begin{aligned} \mathcal{S}\left[1 - p\tau\sum_{k=0}^{\infty}\frac{(-1)^k\tau^{2k}}{1 - r\tau^{2k+1}}\right] &= \sum_{k=0}^{\infty}\frac{\tau^{2k}(C_k - \tau^2 C_{k+1})}{1 - r\tau^{2k+1}} \\ &= \frac{1}{1 - r\tau} - pr(1 - \tau^2)\sum_{k=1}^{\infty}\frac{\tau^{8k-1}}{(1 = r\tau^{4k-1})(1 = r\tau^{4k})(1 = r\tau^{4k+1})} . \end{aligned} \tag{3.35}$$

On the other hand, the recursion (3.22c) allows us to determine the moments a_k

$$\begin{aligned} a_k(1 - r\tau^{2k+2}) &= \frac{p}{2}(1 - \tau^2)\frac{(-1)^k\tau^{2k+1}\mathcal{S}}{(1 - r\tau^{2k+1})(1 - r\tau^{2k+3})} + \frac{\tau^{2k}(1 + r\tau^{2k+1})C_k}{4(1 - r\tau^{2k+1})} \\ &\quad - \frac{r}{2}(1 - \tau^2)\frac{\tau^{4k+3}C_{k+1}}{(1 - r\tau^{2k+1})(1 - r\tau^{2k+3})} - \frac{\tau^{2k+4}(1 + r\tau^{2k+3})C_{k+2}}{4(1 - r\tau^{2k+3})} . \end{aligned} \tag{3.36}$$

A rather lengthy calculation then gives the final outcome

$$\begin{aligned} \chi = 2\beta \Bigg\{ & \frac{2r}{1 - r\tau} - 1 - 2p^2r^2(1 - \tau^2)\sum_{k=1}^{\infty}\frac{\tau^{12k-1}}{(1 - r\tau^{4k-1})(1 - r\tau^{4k})^2(1 - r\tau^{4k+1})} \\ & + 2p\,\frac{\left[\frac{1}{1-r\tau} - pr(1 - \tau^2)\sum_{k=1}^{\infty}\frac{\tau^{8k-1}}{(1-r\tau^{4k-1})(1-r\tau^{4k})(1-r\tau^{4k+1})}\right]^2}{1 - p\sum_{k=0}^{\infty}\frac{(-1)^k\tau^{2k+1}}{1-r\tau^{2k+1}}} \Bigg\} \end{aligned} \tag{3.37}$$

or equivalently

$$\begin{aligned} \chi = 2\beta \Bigg\{ & \frac{2r}{1 - r\tau} - 1 - \frac{2p^2r^2}{(1 - \tau)^2}\sum_{a=0}^{\infty}\frac{r^a\tau^{3a+11}\left[1 - \tau^{2a+4} - (a + 2)(1 - \tau^2)\tau^{a+1}\right]}{1 - \tau^{4a+12}} \\ & + 2p\,\frac{\left[\frac{1}{1-r\tau} - \frac{pr}{1-\tau}\sum_{a=0}^{\infty}\frac{r^a\tau^{3a+7}(1-\tau^{a+1})(1-\tau^{a+2})}{1-\tau^{4a+8}}\right]^2}{1 - p\sum_{a=0}^{\infty}\frac{r^a\tau^{a+1}}{1+\tau^{2a+2}}} \Bigg\} . \end{aligned} \tag{3.38}$$

The general results (3.37) and (3.38) reduce to the susceptibility $\chi = 2\beta(1+\tau)/(1-\tau)$ of two independent Ising chains in the absence of impurities ($p = 0$).

Another limiting case of interest is that of a non-diluted ladder ($p = 1$), where every transversal coupling is non-zero. Consider the slightly more general situation where the transversal couplings assume the values $K_n = \pm\infty$, with respective probabilities q and $1-q$. It follows from (3.24) that the susceptibility of the latter model reads

$$\chi = 8\beta q \left\langle \frac{z_n^{(2)}}{1+z_n^{(0)}} \right\rangle_{K_n=+\infty} = 4\beta q \left(1 + \frac{4q\tau\mathcal{S}}{1+\tau^2}\right). \tag{3.39}$$

Moreover, the constrained average $\mathcal{S}$ is readily obtained from (3.29), which yields

$$\mathcal{S} = 1 + \frac{2q\tau}{1+\tau^2}\mathcal{S} \tag{3.40}$$

so that

$$\mathcal{S} = \frac{1+\tau^2}{1-2q\tau+\tau^2}. \tag{3.41}$$

We are thus left with

$$\chi = 4\beta q \, \frac{1+2q\tau+\tau^2}{1-2q\tau+\tau^2}. \tag{3.42}$$

This expression for $q = \frac{1}{2}$ coincides with (3.37), (3.38) for $p = 1$, namely $\chi = 2\beta(1+\tau+\tau^2)/(1-\tau+\tau^2)$.

3.4. The linear susceptibility at low temperature

The zero-temperature behaviour of the susceptibility can be derived from the result (3.37) by replacing the sums over the integer k by integrals over the continuous variable $2k\mu$. We thus obtain

$$\chi \approx \beta \, \frac{47-10p-p^2}{6p}. \tag{3.43}$$

This expression has the same $1/p$-divergence in the small-p limit as the result (2.45) concerning the random-field model. More generally, the scaling analysis of section 2 applies. In the scaling regime, and again with the notation (2.20) for x, the sums in (3.38) can be turned into integrals over the continuous variable $t = kp$. We thus obtain a one-variable scaling law of the form (2.41) for χ, where the scaling function $S(x)$ now reads

$$S(x) = \frac{4}{1+x} - \frac{4}{x^2}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t(1+2x)}\, \frac{\sinh(tx)-tx}{\sinh(2tx)}$$
$$+ 4\frac{\left[\dfrac{1}{1+x} - \dfrac{1}{x}\displaystyle\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t(1+2x)}\, \frac{\cosh(tx)-1}{\sinh(2tx)}\right]^2}{1 - \dfrac{1}{2}\displaystyle\int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t}\, \frac{1}{\cosh(tx)}}. \tag{3.44}$$

This result can again be expressed in terms of the digamma function and of its first derivative, albeit in a nonlinear way, namely

$$S(x) = \frac{1}{x^3}\left[\frac{1}{2}\psi'\left(\frac{1}{4x}\right) + \psi\left(\frac{1}{4x}+\frac{3}{4}\right) - \psi\left(\frac{1}{4x}+\frac{1}{4}\right) - 4x(1+x)\right] + \frac{\frac{1}{4x^4}\left[2\psi\left(\frac{1}{4x}\right) - \psi\left(\frac{1}{4x}+\frac{3}{4}\right) - \psi\left(\frac{1}{4x}+\frac{1}{4}\right) + 4x\right]^2}{1-\frac{1}{4x}\left[\psi\left(\frac{1}{4x}+\frac{3}{4}\right) - \psi\left(\frac{1}{4x}+\frac{1}{4}\right)\right]}. \tag{3.45}$$

This expression is considerably more complicated than its analogue (2.43) concerning the random-field model. The point to be stressed is that $S(x)$ is no longer linear in the Eulerian functions in the present case. This observation virtually excludes any possibility of having a closed-form expression for the generating function of the correlations in the low-temperature scaling regime.

The scaling function $S(x)$ has the following behaviour for small and large values of its argument

$$S(x) = \tfrac{47}{6} - \tfrac{376}{15}x^2 + \cdots \qquad x \to 0 \tag{3.46a}$$

$$S(x) = \frac{4}{x} + \frac{\pi^2/12 + 2\pi - 12\ln 2}{x^3} + \cdots \qquad x \to \infty. \tag{3.46b}$$

The value $S(0) = \frac{47}{6}$ agrees with the result (3.43), whereas the fall-off $S(x) \approx 4/x$ matches the crossover to the susceptibility of two uncoupled ferromagnetic chains.

Figure 4 shows a plot of the scaling function $S(x)$.

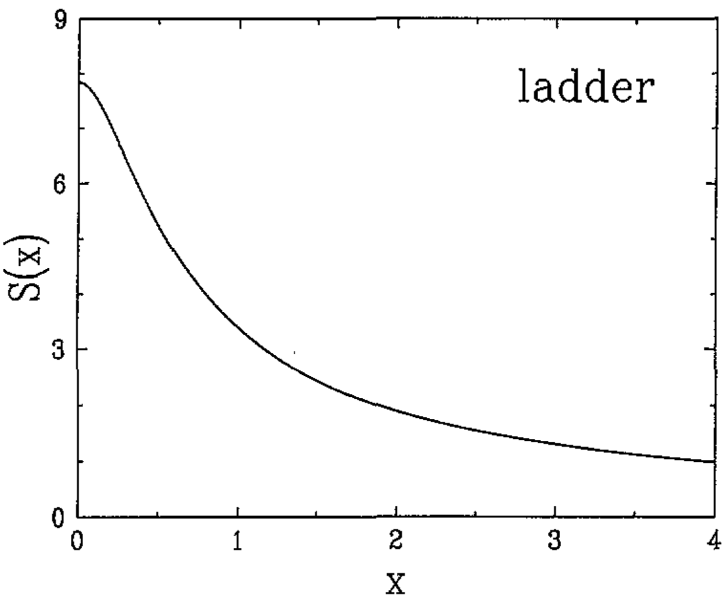

Figure 4. Plot of the scaling function $S(x)$ of the linear susceptibility χ of the ladder spin glass with infinitely strong random exchange couplings.

4. Discussion

We have investigated the susceptibilities of two examples of 1D disordered and frustrated magnetic models, namely the ferromagnetic Ising chain in a random field and the ladder spin glass, with diluted disorder generated by infinitely strong impurities. In the first example of a chain in a diluted random field, studied first by Grinstein and Mukamel, the non-zero fields pin the impurity spins at any temperature. The chain is thus split into independent finite pieces, so that its free energy in an external field can be evaluated as a weighted sum over clusters. An explicit calculation of the susceptibilities is thus possible. In the second example of a ladder spin glass with infinitely strong random exchange couplings, the pairs of spins $(\sigma_n^{(1)}, \sigma_n^{(2)})$ of the impurity columns are forced to be either parallel or anti-parallel, at any temperature. This kind of frustration is different from the previous one, since only the relative sign $\sigma_n^{(1)}\sigma_n^{(2)}$ of the pairs of impurity spins is fixed, so that the ladder is not split into independent clusters. Indeed the value of $\sigma_n^{(1)}$, say, propagates some dynamical information along the whole system, whence the higher level of difficulty. In this second model, we have obtained exactly the zero-field thermodynamics and the linear susceptibility at finite temperature.

One of the main outcomes of this study concerns the behaviour of the susceptibilities in the scaling regime where the thermal correlation length ξ and the typical distance L between impurities are comparable to each other, and much larger than the lattice spacing. The discussion which follows relies on our detailed analysis of the Grinstein–Mukamel model of a ferromagnetic chain in a random field. We claim that analogous scaling laws, with the same exponents, hold true for any 1D or quasi-1D magnetic model with diluted disorder, whether or not the random fields or exchange couplings are infinitely strong. This assertion is corroborated by the following two facts. The linear susceptibility of the ladder spin glass follows the same scaling law as that of the random-field chain, but with a different, more complicated, functional form for the amplitude $S(x)$. The susceptibility of the random-field Ising chain is also known [14] for a class of diluted continuous distributions of the random fields. Its low-temperature behaviour coincides with that obtained here, again up to an absolute prefactor, which bears the dependence on the distribution of the random fields.

The scaling laws $\chi_{2\ell-1} \approx (\beta/\mu)^{2\ell-1} S_{2\ell-1}(x)$ are a consequence of the more general two-variable scaling behaviour (2.21) of the singular part of the free energy, which involves the magnetic length L_H as a third length scale. At first sight, the low-temperature estimates $\chi \approx \beta/(3p)$ and $\chi_3 \approx 4\beta^3/(15p^3)$, implying $\chi_3 \gg \chi^2$, seem to be in rough qualitative agreement with the case of experimental (3D) spin glasses, or with the predictions of mean-field theory, namely that χ_3 diverges, whereas χ stays finite at the spin-glass transition. On the other hand, the estimates recalled just above are not a peculiarity of the random fixed point of 1D model systems, since they hold throughout the scaling regime, including the limiting situation of the pure ferromagnetic chain, where $\chi \approx \beta/\mu$ and $\chi_3 \approx \beta^3/(2\mu^3)$, so that again $\chi_3 \gg \chi^2$.

This is no paradox. Consider the case of typical critical phenomena, governed by an isolated fixed point, with only two independent critical exponents, η and ν, and with the standard hyperscaling laws. The singular part of the free energy reads

$$F_{\mathrm{sg}} \approx |t|^{d\nu} \Phi\left(H|t|^{-(d+2-\eta)\nu/2}\right) \tag{4.1}$$

with $t = (T - T_C)/T_C$, and d the spatial dimension. The susceptibilities therefore obey the power laws $\chi_{2\ell-1} \sim |t|^{-\gamma_{2\ell-1}}$, with exponents

$$\gamma_{2\ell-1} = \left[(d+2-\eta)\ell - d\right]\nu \qquad \ell \geqslant 1\,. \tag{4.2}$$

The inequality $\chi_3 \gg \chi^2$ is thus rather a general rule than an exception, since $\gamma_3 - 2\gamma = d\nu > 1$ at a continuous phase transition.

As it turns out, the Sherrington–Kirkpatrick mean-field theory of spin glasses is an exception to the scaling laws (4.1) and (4.2), even on the paramagnetic side of the spin-glass transition ($T > T_G$), where there is no trouble whatsoever with replica-symmetry breaking. Indeed the linear susceptibility χ plays a special role in the theory, so that the free energy obeys a modified scaling formula [1–3]

$$F_{sg} \approx -\frac{H^2}{2T_G}\left[1 - t\,\Psi\left(\frac{H}{T_G\,t}\right)\right] \tag{4.3}$$

for both H and t small, with the notation $t = (T - T_G)/T_G$, and where the scaling function $\Psi(x)$ is even and regular around $x = 0$. This implies that the linear susceptibility $\chi(T_G) = 1/T_G$ is finite, whereas only the higher-order susceptibilities diverge, with the exponents $\gamma_{2\ell-1} = 2\ell - 3$ for $\ell \geqslant 2$.

The zero-temperature transition of the pure ferromagnetic Ising chain fits the general scheme (4.1), (4.2) of usual critical phenomena, up to the identification $H \to \beta H$, $F \to \beta F$, $|t| \to \mu$, and with formally $\nu = \eta = 1$.

The explicit results obtained in this work demonstrate that randomly frustrated 1D magnetic systems with *diluted* disorder, unlike mean-field spin glasses, also fit the scaling scheme (4.1) and (4.2) near their zero-temperature fixed point, with the identification mentioned just above, but with p replacing μ. The extra scaling variable $x = 2\mu/p$ describes the crossover inside the scaling regime from the physics of the random system for x small to that of the pure system for x large.

We emphasize that the scaling behaviour of the nonlinear susceptibilities for 1D magnetic systems with a *continuous*, *non-diluted* distribution of the random fields or exchange couplings remains unknown so far. The low-temperature physics in the presence of diluted and non-diluted disorder is indeed very different [4, 9, 14]. For instance in the latter case the specific heat generically vanishes linearly with the temperature, whereas the linear susceptibility reaches a finite zero-temperature limit.

Among other aspects of the present work, let us emphasize the estimate (2.35) of the Lee–Yang singularity. This exponentially small essential singularity in the complex H-plane is very reminiscent of the Lifshitz singularities of, for example, disordered electronic spectra at their band edges (see [4, 23] for reviews). Connections between Lifshitz singularities and the Griffiths singularities of the free energy of some disordered magnetic models in the complex T-plane [24] have already been established [25].

Acknowledgments

It is a pleasure to thank Theo Nieuwenhuizen for very stimulating discussions. Part of this work has been done by EA during his *stage de DEA de Physique Quantique* (Paris).

References

[1] Mézard M, Parisi G and Virasoro M A 1987 *Spin-Glass Theory and Beyond* (Singapore: World Scientific)
[2] Fischer K H and Hertz J A 1991 *Spin Glasses* (Cambridge: Cambridge University Press)
[3] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
Kirkpatrick S and Sherrington D 1978 *Phys. Rev.* B **17** 4384

[4] Luck J M 1992 *Systèmes Désordonnés Unidimensionnels* (Collection Aléa-Saclay)
[5] Crisanti A, Paladin G and Vulpiani A 1992 *Products of Random Matrices in Statistical Physics* (Berlin: Springer)
[6] Grinstein G and Mukamel D 1983 *Phys. Rev.* B **27** 4503
[7] Nieuwenhuizen Th M and Luck J M 1986 *J. Phys. A: Math. Gen.* **19** 1207
[8] Funke M, Nieuwenhuizen Th M and Trimper S 1989 *J. Phys. A: Math. Gen.* **22** 5097
[9] Luck J M, Funke M and Nieuwenhuizen Th M 1991 *J. Phys. A: Math. Gen.* **24** 4155
[10] Derrida B, Vannimenus J and Pomeau Y 1978 *J. Phys. C: Solid State Phys.* **11** 4749
[11] Derrida B and Hilhorst H J 1983 *J. Phys. A: Math. Gen.* **16** 2641
[12] Samaj L and Markos P 1991 *J. Phys. A: Math. Gen.* **24** 1319
[13] Mello P A and Robledo A 1993 *Physica* **199A** 363
[14] Luck J M and Nieuwenhuizen Th M 1989 *J. Phys. A: Math. Gen.* **22** 2151
[15] Normand J M, Mehta M L and Orland H 1985 *J. Phys. A: Math. Gen.* **18** 621
[16] Clerc J P, Giraud G, Laugier J M and Luck J M 1990 *Adv. Phys.* **39** 191
[17] McCoy B M and Wu T T 1973 *The Two-Dimensional Ising Model* (Cambridge, MA: Harvard University Press) and the references therein
[18] Nieuwenhuizen Th M and Orland H 1989 *Phys. Rev.* B **40** 5094
[19] Luck J M 1993 *J. Stat. Phys.* **72** 417
[20] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)
[21] Lee T D and Yang C N 1952 *Phys. Rev.* **87** 410
[22] Itzkyson C and Drouffe J M 1989 *Statistical Field Theory* vol 2 (Cambridge: Cambridge University Press)
[23] Nieuwenhuizen Th M 1990 *Physica* **167A** 43
[24] Griffiths R B 1969 *Phys. Rev. Lett.* **23** 17
[25] Nieuwenhuizen Th M 1989 *Phys. Rev. Lett.* **63** 1760